

Central exponents of linear stochastic differential-algebraic equations of index 1

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Our aim in this paper is to establish a concept of central exponents of a stochastic differential-algebraic equation (SDAE) of index 1. For this purpose, we introduce the inherent regular stochastic differential equation (SDE) associated with a SDAE of index 1. Then, the central exponents are defined samplewise via the induced two-parameter stochastic flow generated by the inherent regular SDE. We prove that the central exponents are nonrandom and greater than or equal to the Lyapunov exponents.

Keywords: Lyapunov exponents; central exponents; differential-algebraic equations of index 1; stochastic differential equations; stochastic differential-algebraic equations of index 1; two-parameter stochastic flow; induced two-parameter stochastic flow.

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1. Introduction

In science and practical applications, there are numerous problems, such as the problem of description of dynamical systems and electric circuits or problems in cybernetics, requiring investigation of differential-algebraic systems (singular systems) with noise. The mathematical model of such systems leads to stochastic differential-algebraic equations (SDAEs). This kind of equation can be considered as a generalization of differential-algebraic equations (DAEs) and stochastic differential equations (SDEs).

Investigation of DAE and SDE was carried out intensively by many researchers around the world. In particular, the stability theory and most concepts related to the Lyapunov theory of ordinary differential equations are generalized to DAEs and SDEs (see e.g., [3, 4, 15, 19, 20, 27], [1, 14, 16–18, 22] and the references therein, respectively). Under certain conditions, a SDAE can be transformed into a system consisting of a SDE and an algebraic equation, which allows us to use methods and results of the theory of SDEs to investigate SDAEs. However, the investigation of

SDAEs is an ongoing area of research, and the concepts, qualitative properties, and numerical treatment of SDAEs are not as well-established as those of DAEs and SDEs. We here mention [6–8, 11–13, 29, 30] among others. Some of these works have dealt with the theory of Lyapunov for SDAEs. More precisely, the authors of [6, 7] investigated the stability of SDAE by using the method of Lyapunov functions. Meanwhile, [11, 13] was devoted to the concepts of Lyapunov exponents, Lyapunov spectrum, adjoint equation and the Lyapunov regularity of linear SDAE of index 1. Our aim in this paper is to continue establishing the theory of Lyapunov exponents by extending the notion of the central exponent to the class of nonautonomous linear SDAE of index 1. The importance and motivation for study of central exponents are as follows:

It is well known for ordinary differential equations that, besides the Lyapunov exponents (see [5, 21]), the central exponents, introduced by Vinograd in 1957 (see [28]), also serve as qualitative and quantitative characteristics of the equations under consideration. Without the regular property, the negativity of the highest Lyapunov exponent does not always imply the stability of nonlinearly perturbed systems. However, if the top central exponent is negative, then the stability of nonlinearly perturbed systems is guaranteed.

The theory of Lyapunov exponents, when applied to the framework of ergodic theory, leads to a whole new field of research: the theory of random dynamical systems (see [2]). This approach enables us to use the tools of random dynamical systems to investigate the Lyapunov exponent for autonomous SDEs since there is a one-to-one correspondence between them. However, the nonautonomous SDEs do not fit into the framework of the theory of random dynamical systems. In this case, based on the classical theory of Lyapunov, which was further developed by Millionshchikov [24–26], and the theory of two-parameter stochastic flows by Kunita [18], the author of [9] introduced the concept of Lyapunov exponents and central exponent for linear SDEs and investigated the Lyapunov regularity of SDEs. Furthermore, the author of [9] proved that the top central exponent, rather than the top Lyapunov exponent, is responsible for the stability of a linear SDE with respect to small linear perturbations (see [9, Theorem 5.9]).

This paper is organized as follows: in the following section, we recall some basic notions about two-parameter stochastic flows, the Lyapunov exponents and central exponents of nonautonomous linear SDE, the two-parameter stochastic flows induced by linear SDE, the Lyapunov exponents of DAE of index 1. In Sec. 3, we introduce the concept of central exponents of nonautonomous linear SDAE of index 1, prove that they are nonrandom and compare them with Lyapunov exponents.

The following notations are used throughout the paper: \mathcal{G}_k is the Grassmannian manifold of all k -dimensional subspaces of \mathbb{R}^n . For a linear subspace U of \mathbb{R}^n , we denote by U_* the subset of all nonvanishing vectors of U . Let $\mathbb{T}|_U$ denote the restriction of the operator \mathbb{T} on U with operator norm $\|\mathbb{T}|_U\| = \sup_{x \in U_*} \frac{\|\mathbb{T}(x)\|}{\|x\|}$.

2. Preliminary

2.1. Lyapunov exponents and central exponents of nonautonomous linear stochastic differential equation

Consider a nonautonomous linear SDE

$$dx(t) = F_0(t)x(t)dt + \sum_{j=1}^d F_j(t)x(t)dW_t^j, \quad t \in \mathbb{R}^+, \quad (2.1)$$

where the matrix-valued functions $F_j : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$, $j = 0, \dots, d$, are continuous and $W_t := (W_t^1, W_t^2, \dots, W_t^d)$ is a d -dimensional standard Wiener process given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see [1]). In this section, we review the notions of the Lyapunov exponents and central exponents for SDE (2.1). These characteristic exponents are defined via two-parameter stochastic flow introduced in [18]. First, we recall the concept of two-parameter stochastic flow on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the state space \mathbb{R}^n . A *two-parameter stochastic flow of diffeomorphisms of \mathbb{R}^n* on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of \mathbb{R}^n -valued random variables $\Phi_{s,t}(\cdot)x : \Omega \rightarrow \mathbb{R}^n$, where $s, t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, which satisfies the following properties for any ω from a subset $\Omega' \subset \Omega$ of full \mathbb{P} -measure:

- (i) the map $(s, t, x) \mapsto \Phi_{s,t}(\omega)x$ is continuous;
- (ii) $\Phi_{s,t}(\omega) = \Phi_{u,t}(\omega) \circ \Phi_{s,u}(\omega)$ holds for all $s, t, u \in \mathbb{R}^+$, where \circ denotes the composition of maps;
- (iii) $\Phi_{s,s}(\omega)$ is the identity map for all $s \in \mathbb{R}^+$;
- (iv) the map $\Phi_{s,t}(\omega) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an onto homeomorphism for all $s, t \in \mathbb{R}^+$;
- (v) $\Phi_{s,t}(\omega)x$ is differentiable with respect to $x \in \mathbb{R}^n$ for all $s, t \in \mathbb{R}^+$ and the derivative is continuous in s, t, x .

Further, if the maps $\Phi_{s,t}(\omega)$, $s, t \in \mathbb{R}^+$ are almost surely linear then the flow $\Phi_{s,t}(\omega)$ is called a *two-parameter stochastic flow of linear operators of \mathbb{R}^n* .

Since we may restrict our consideration to the set Ω' of full \mathbb{P} -measure, from now on we shall assume that the above properties (i)–(v) are satisfied for all $\omega \in \Omega$. The linear SDE (2.1) generates a two-parameter stochastic flow of linear operators of \mathbb{R}^n (see Kunita [18]). Note that fixing an $\omega \in \Omega$ the two-parameter stochastic flow $\Phi_{s,t}(\omega)$ is an analog of the Cauchy operator of a linear system of differential equations. Hence, we are able to define Lyapunov exponents and central exponents for this family $\Phi_{s,t}(\omega)$ by using the classical theory of Lyapunov exponents. Millionshchikov [24–26] had discovered that there are several equivalent definitions for the Lyapunov exponents and central exponents. We now follow [9, 10] to define the Lyapunov exponents and central exponents of (2.1).

Suppose that $\Phi_{s,t}(\omega)$ is a two-parameter stochastic flow of linear operators of \mathbb{R}^n generated by SDE (2.1). Denote by \mathcal{G}_k the Grassmannian manifold of all k -dimensional subspaces of \mathbb{R}^n . The extended-real numbers

$$\lambda_k(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \sup_{x \in V} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|\Phi_{0,t}(\omega)x\|, \quad k = 1, \dots, n,$$

are called *Lyapunov exponents* of SDE (2.1). The set $\{\lambda_1(\omega), \dots, \lambda_n(\omega)\}$ is called *Lyapunov spectrum* of SDE (2.1).

The extended-real numbers

$$\Omega_k(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \inf_{T \in R^+} \limsup_{\substack{m \in \mathbb{N} \\ m \rightarrow \infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Phi_{iT, (i+1)T}(\omega)|_{\Phi_{0,iT}(\omega)V}\|, \\ k = 1, 2, \dots, n$$

are called the *central exponents* of SDE (2.1).

The above concepts of Lyapunov exponents and central exponents of SDE are defined for the sample path as in the classical deterministic case. However, the author of [9, 10] used some deep probabilistic arguments to prove that they are nonrandom and, in general,

$$\Omega_k \geq \lambda_k \quad \text{for } k = 1, 2, \dots, n.$$

Furthermore, the author of [9] showed that the top central exponent Ω_1 of SDE (2.1) is upper semicontinuous with respect to linear perturbations, and if $\Omega_1 < 0$ then the small linearly perturbed SDE is stable with probability one (see [9, Theorem 5.9]).

2.2. Lyapunov exponents of stochastic differential-algebraic equation of index 1

Consider a nonautonomous linear SDAE

$$A(t)dx(t) + B(t)x(t)dt + \sum_{j=1}^d B_j(t)x(t)dW_t^j = 0, \quad t \in \mathbb{R}^+, \quad (2.2)$$

where the matrix-valued functions $A, B, B_j : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times n}$ are continuous and bounded functions on \mathbb{R}^+ , (W_t) is an m -dimensional Wiener process given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the standard filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$. Suppose that the leading coefficient $A(t)$ is singular and $\text{rank } A(t) = r$, where r is a fixed integer with $r < n$. The following DAE:

$$A(t)dx(t) + B(t)x(t)dt = 0, \quad t \in \mathbb{R}^+ \quad (2.3)$$

is called the *deterministic part* of (2.2). The DAE (2.3) is called to be of *index 1* if there exists a smooth projector $Q \in C^1(\mathbb{R}^+, \mathbb{R}^{n \times n})$ onto $\ker A(t)$ such that $A_1(t) := A(t) + B_0(t)Q(t)$ is nonsingular on \mathbb{R}^+ , where $P(t) := I - Q(t)$; $B_0(t) := B(t) - A(t)P'(t)$ (see [12, 15]).

Recall from [12] that the SDAE (2.2) is said to be of *index 1* if

- (i) the deterministic part of (2.2) is a DAE of index 1;
- (ii) $\text{im } B_j(t)x \subset \text{im } A(t)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and $j = 1, \dots, d$.

Since the SDAE is of index 1, under the projector Q onto $\ker A(t)$ and the respect projector P along $\ker A(t)$, we transform (2.2) into a SDE (see [12, 13])

$$du(t) = (P' - PA_1^{-1}B_0)u(t)dt + \sum_{j=1}^d F_j(t)u(t)dW_t^j, \quad t \in \mathbb{R}^+, \quad (2.4)$$

where

$$F_j(t) := -A_1^{-1}(t)B_j(t)P_{\text{can}}(t), \quad j = 1, \dots, d,$$

$Q_{\text{can}} := QA_1^{-1}B_0$ and $P_{\text{can}} := I - Q_{\text{can}}$. Note that Q_{can} is a projector onto $\ker A$ which is independent of the choice of the smooth projector Q (see [12, 13]).

The SDE (2.4) is called the *inherent SDE* of (2.2) (under P). Let $\Phi_{s,t}(\omega)$ is a two-parameter stochastic flow generated by (2.4) (see [18]). Then,

$$x(t) = P_{\text{can}}(t)\Phi_{s,t}(\omega)P(s)x_0, \quad x_0 \in \mathbb{R}^n,$$

is the solution of (2.2) satisfying the initial condition $x(s) - x_0 \in \ker A(s)$, $t \geq s \geq 0$. The following family of linear operators:

$$\Psi_{s,t}(\omega) := P_{\text{can}}(t)\Phi_{s,t}(\omega)P(s) \quad (2.5)$$

is called *the induced two-parameter flow of (2.2)* (see [13]). It was proved in [13] that $\Psi_{s,t}(\omega)$ does not depend on the choice of the projector Q . Thanks to the properties of two-parameter flow of SDE, the property of projectors and the fact that $\text{im } P(t)$ is an invariant subspace of the inherent regular SDE (2.4) (see [12, Remark 3.12]), the authors of [11] proved that the induced two-parameter flow satisfies the following properties for any projector $P(t)$ along $\ker A(t)$ and $s, t \in \mathbb{R}^+$ (see [11, Proposition 4.2]):

$$\Psi_{0,t}(\omega) = \Psi_{s,t}(\omega)\Psi_{0,s}(\omega). \quad (2.6)$$

From [13], the *Lyapunov exponents* of the SDAE (2.2) of index 1 are defined by

$$\lambda_k(\omega) = \inf_{V \in \mathcal{G}_{n-k+1}} \sup_{x \in V} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|\Psi_{0,t}(\omega)x\|, \quad k = 1, \dots, r,$$

where \mathcal{G}_k denotes the Grassmannian manifold of all k -dimensional subspaces of \mathbb{R}^n . It was shown in [13] that the Lyapunov exponents $\lambda_k(\omega)$, $k = 1, \dots, r$ are nonrandom. The set consisting of the Lyapunov exponents

$$\lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots \geq \lambda_r(\omega)$$

is called the *Lyapunov spectrum* of (2.2).

3. Central Exponents of Stochastic Differential-Algebraic Equation of Index 1

In this section, we introduce a concept of central exponents for SDAE (2.2) of index 1. Let

$$\Psi_{s,t}(\omega) = P_{\text{can}}(t)\Phi_{s,t}(\omega)P(s)$$

be the induced two-parameter flow of (2.2), where $\Phi_{s,t}(\omega)$ is the two-parameter stochastic flow generated by inherent SDE (2.4).

Definition 3.1. Suppose that (2.2) is an SDAE of index 1 and $\Psi_{s,t}(\omega)$ is the induced two-parameter flow of (2.2). Then

$$\Lambda_k(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \inf_{T \in \mathbb{R}^+} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|, \\ k = 1, 2, \dots, r \quad (3.1)$$

are called the *central exponents* of (2.2).

Remark 3.1. For $k > r$, since $\dim \text{Ker } A(0) = n - r$, there exists $V \in \mathcal{G}_{n-k+1}$, $V \subset \text{Ker } A(0)$. Hence, $\Psi_{0,t}(\omega)V = 0$ for $t \geq 0$. Consequently, we can define $\Lambda_k(\omega) = -\infty$ for $k = r + 1, \dots, n$.

We now state the first main result of the paper about the nonrandomness of the central Lyapunov exponents of an SDAE of index 1.

Theorem 3.1. *The central exponents $\Lambda_k(\omega)$ of SDAE (2.2) of index 1 are not random.*

To prove Theorem 3.1 on the nonrandomness of central exponents of SDAE (2.2), we follow the idea in [10]. However, unlike the two-parameter flow $\Phi_{s,t}(\omega)$ of SDE considered in [10], the induced two-parameter flow $\Psi_{s,t}(\omega)$ does not consist of bijective mappings of \mathbb{R}^n . Here, to overcome these difficulties, we employ a similar approach developed in [13] for proving the nonrandomness of Lyapunov exponents of SDAE. For this purpose, we need the following lemmas. First, for convenience, we recall [13, Lemma 16]. For any $k = 1, \dots, r$ and $s \geq 0$, we put

$$\mathcal{G}(s)'_{n-k+1} := \{V \in \mathcal{G}_{n-k+1} : V = \ker A(s) \oplus W, W \subset \text{im } P(s)\}.$$

Lemma 3.1 ([13, Lemma 16]). *For any fixed $\omega \in \Omega$, $t > 0$, the map*

$$h_t(\omega) : \mathcal{G}(0)'_{n-k+1} \rightarrow \mathcal{G}(t)'_{n-k+1},$$

defined by the formula

$$\mathcal{G}(0)'_{n-k+1} \ni \ker A(0) \oplus W \mapsto \ker A(t) \oplus \Phi_{0,t}(\omega)W \in \mathcal{G}(t)'_{n-k+1}$$

is bijective.

Note that the result above remains valid if we replace the initial time 0 with any time $s \in \mathbb{R}^+$.

Lemma 3.2. *For $k \in \{1, \dots, r\}$, $t \geq s \geq 0$ and $V \in \mathcal{G}(0)'_{n-k+1}$, let $U := h_s(\omega)V$. Then $U \in \mathcal{G}(s)'_{n-k+1}$ and*

$$\Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)P(0)V) = \Psi_{s,t}(\omega)U. \quad (3.2)$$

Proof. Suppose that $V \in \mathcal{G}(0)'_{n-k+1}$, that is, $V = \ker A(0) \oplus W, W \subset \text{im } P(0)$. Let $U = h_s(\omega)V$. Then, by Lemma 3.1, $U = \ker A(s) \oplus \Phi_{0,s}(\omega)W \in \mathcal{G}(s)'_{n-k+1}$. Moreover,

$$\begin{aligned}
 \Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)P(0)V) &= \Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)P(0)(\ker A(0) \oplus W)) \\
 &= \Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)P(0)W) \\
 &= \Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)W) \\
 &= \Psi_{s,t}(\omega)(\ker A(s) \oplus \Phi_{0,s}(\omega)W) \\
 &= \Psi_{s,t}(\omega)(h_s(\omega)V) = \Psi_{s,t}(\omega)U. \quad \square
 \end{aligned}$$

Lemma 3.3. For $k \in \{1, \dots, r\}$, $T > 0$ and $V \in \mathcal{G}_{n-k+1}$ there exists $U \in \mathcal{G}(0)'_{n-k+1}$ such that

$$\begin{aligned}
 &\limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\
 &\geq \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|.
 \end{aligned}$$

Proof. For $V \in \mathcal{G}_{n-k+1}$, we will take $U \in \mathcal{G}(0)'_{n-k+1}$ as in [13, Lemma 17]. More precisely, we denote

$$V_1 := V \cap \ker A(0),$$

and let $\{e_1, \dots, e_s\}$ be a basis of V_1 . Here, we use the convention that if $V_1 = \{0\}$ we let $s = 0$. We add linearly independent vectors $\{x_1, \dots, x_{n-k+1-s}\}$ of V to the basis $\{e_1, \dots, e_s\}$ of V_1 to get a basis $\{e_1, \dots, e_s, x_1, \dots, x_{n-k+1-s}\}$ of V . Since $\text{im } P(0) \oplus \ker A(0) = \mathbb{R}^n$, the vectors x_i can be decomposed uniquely into the form

$$x_i = y_i + z_i, \quad y_i \in \text{im } P(0), \quad z_i \in \ker A(0), \quad i = 1, \dots, n - k + 1 - s.$$

Denote by V_2 the subspace of \mathbb{R}^n spanned by the vectors $\{x_1, \dots, x_{n-k+1-s}\}$ and by V_3 the subspace spanned by the vectors $\{y_1, \dots, y_{n-k+1-s}\}$. Obviously, $V = V_1 \oplus V_2$ and $W \subset \text{im } P(0)$. Since $V_1 = V \cap \ker A(0)$, it is easily seen that the vectors $y_1, \dots, y_{n-k+1-s}$ are linearly independent, and they form a basis of V_3 . Hence, $\dim V_3 = n - k + 1 - s = \dim V_2$. From $k \in \{1, \dots, r\}$ and

$$s = \dim V_1 \leq \dim \ker A(0) = n - r,$$

we have

$$\dim V_3 = n - k + 1 - s \geq (n - k + 1) - (n - r) = r - k + 1 > 0.$$

Therefore, there exists a subspace W of V_3 with $\dim W = r - k + 1$. Now, put

$$U := \ker A(0) \oplus W.$$

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Then, clearly $U \in \mathcal{G}(0)'_{n-k+1}$. On the other hand, by the fact that $\Psi_{0,s}(\omega)z = 0$ for any $z \in \ker A(0)$ and $s \geq 0$, we have

$$\{\Psi_{0,s}(\omega)U\} = \{\Psi_{0,s}(\omega)W\} \subset \{\Psi_{0,s}(\omega)V_3\} = \{\Psi_{0,s}(\omega)V_2\} = \{\Psi_{0,s}(\omega)V\},$$

which implies that for any $t \geq s \geq 0$,

$$\|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)V}\| \geq \|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)U}\|.$$

Therefore, for any $T > 0$ and $m \in \mathbb{N}_*$, $i = 0, 1, \dots, m-1$

$$\|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \geq \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|.$$

Hence,

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ & \geq \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|. \end{aligned} \quad \square$$

Lemma 3.4. *For $k \in \{1, \dots, r\}$, $T > 0$, we have*

$$\begin{aligned} & \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ & = \inf_{V \in \mathcal{G}(0)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|. \end{aligned}$$

Proof. Take and fix an arbitrary $V \in \mathcal{G}_{n-k+1}$ and choose $U \in \mathcal{G}(0)'_{n-k+1}$ as in Lemma 3.3. Then,

$$\begin{aligned} & \inf_{V' \in \mathcal{G}(0)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V'}\| \\ & \leq \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\| \\ & \leq \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|. \end{aligned}$$

Since V is fixed but arbitrary in \mathcal{G}_{n-k+1} it follows that

$$\begin{aligned} & \inf_{V \in \mathcal{G}(0)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ & \leq \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|. \end{aligned}$$

On the other hand, from $\mathcal{G}(0)'_{n-k+1} \subset \mathcal{G}_{n-k+1}$ we get the inverse inequality. The proof of the lemma is completed. \square

We will use the ideas and techniques developed in [10, 13, 23] to demonstrate that the central exponents of SDAEs are nonrandom. For this purpose, we recall the following result developed by Millionshchikov in [23, Lemma 2].

Lemma 3.5. *For any $k \in \{1, \dots, r\}$ and $\omega \in \Omega$ the central exponents $\Lambda_k(\omega)$ defined as in (3.1) satisfy*

$$\Lambda_k(\omega) = \lim_{T \rightarrow \infty} \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|.$$

In what follows, let $\mathcal{F}_s^t := \sigma(W_u - W_s, t \geq u \geq s \geq 0)$. Note that $\mathcal{F}_t = \mathcal{F}_0^t$, where $\mathcal{F}_t := \sigma(W_s, t \geq s \geq 0), t \in \mathbb{R}^+$, is the natural filtration of the Brownian motion $(W_t)_{t \geq 0}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 3.6. *For any $k \in \{1, 2, \dots, r\}$, $N \in \mathbb{N}$, $T > 0$, we define the function $\Lambda_{k,N,T}(\omega)$ as follows:*

$$\Lambda_{k,N,T}(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)V}\|. \quad (3.3)$$

Then, $\Lambda_{k,N,T} : \Omega \rightarrow \bar{\mathbb{R}}$ is \mathcal{F}_{NT}^∞ -measurable, where $\mathcal{F}_{NT}^\infty := \sigma(W_t - W_{NT}, t \geq NT)$.

Proof. Let us fix arbitrary an $k \in \{1, 2, \dots, r\}$, $N \in \mathbb{N}$ and $T > 0$. Then, for any $m \in \mathbb{N}_*, m > N, q \in \mathbb{N}_*$, $V \in \mathcal{G}_{n-k+1}$ and $\omega \in \Omega$, we define

$$a_m(V, \omega) := \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)V}\| \quad (3.4)$$

and

$$f(m, q)(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \max_{t \in \{0, \dots, q\}} a_{m+t}(V, \omega). \quad (3.5)$$

We have

$$\begin{aligned} f(m, q+1)(\omega) &= \inf_{V \in \mathcal{G}_{n-k+1}} \max_{t \in \{0, 1, \dots, q+1\}} a_{m+t}(V, \omega) \\ &\geq \inf_{V \in \mathcal{G}_{n-k+1}} \max_{t \in \{0, 1, \dots, q\}} a_{m+t}(V, \omega) \\ &= f(m, q)(\omega). \end{aligned}$$

Therefore, the sequence $\{f(m, q)(\omega)\}_q$ is non-decreasing. Consequently, there exists the limit

$$\lim_{q \rightarrow \infty} f(m, q)(\omega) =: g(m)(\omega). \quad (3.6)$$

Similarly, the sequence $\{g(m)(\omega)\}_m$ is non-increasing. This follows that there exists the limit $\lim_{m \rightarrow \infty} g(m)(\omega)$. Thus,

$$\begin{aligned} \lim_{m \rightarrow \infty} g(m)(\omega) &\stackrel{(3.6)}{=} \lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} f(m, q)(\omega) \\ &\stackrel{(3.5)}{=} \lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} \inf_{V \in \mathcal{G}_{n-k+1}} \max_{t \in \{0, \dots, q\}} a_{m+t}(V, \omega). \end{aligned} \quad (3.7)$$

Since $\Phi_{s,t}(\omega)$ is the two-parameter stochastic flow of linear operators of \mathbb{R}^n generated by the inherent regular SDE (2.4) and $\Psi_{s,t}(\omega) = P_{\text{can}}(t)\Phi_{s,t}(\omega)P(s)$, the function

$$\begin{aligned} [0, 1, \dots, q] \times \mathcal{G}_{n-k+1} \times \Omega &\rightarrow \mathbb{R} \\ (t, V, \omega) &\mapsto a_{m+t}(V, \omega) \end{aligned}$$

is \mathcal{F}_{NT}^∞ -measurable and continuous in (t, V) . This implies that $\inf_{V \in \mathcal{G}_{n-k+1}} \times \max_{t \in \{0, 1, \dots, q\}} a_{m+t}(V, \omega)$ is $\bar{\mathbb{R}}$ -valued random variable and is \mathcal{F}_{NT}^∞ -measurable, since \mathcal{G}_{n-k+1} is compact. By (3.7), so is $\lim_{m \rightarrow \infty} g(m)(\cdot)$. Moreover, also by the compactness of the Grassmannian manifold \mathcal{G}_{n-k+1} , the monotonicity property of the functions above and by the fact that

$$\limsup_{m \rightarrow \infty} x_m = \lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} \max_{t \in \{0, \dots, q\}} x_{m+t},$$

for any sequence of real numbers x_m , we obtain

$$\begin{aligned} \Lambda_{k,N,T}(\omega) &\stackrel{(3.3)}{=} \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{NT, iT}(\omega)V}\| \\ &\stackrel{(3.4)}{=} \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} a_m(V, \omega) \\ &= \inf_{V \in \mathcal{G}_{n-k+1}} \lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} \max_{t \in \{0, \dots, q\}} a_{m+t}(V, \omega) \\ &= \lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} \inf_{V \in \mathcal{G}_{n-k+1}} \max_{t \in \{0, \dots, q\}} a_{m+t}(V, \omega) \\ &\stackrel{(3.7)}{=} \lim_{m \rightarrow \infty} g(m)(\omega). \end{aligned} \quad (3.8)$$

Consequently, $\Lambda_{k,N,T}(\omega)$ is $\bar{\mathbb{R}}$ -valued random variable and is \mathcal{F}_{NT}^∞ -measurable. The lemma is proved. \square

Now, we turn to a proof of Theorem 3.1.

Proof of Theorem 3.1. Let us fix $k \in \{1, 2, \dots, r\}$. For each $T > 0$, let

$$\Lambda_{k,T}(\omega) := \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT, (i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|.$$

First, we will show that $\Lambda_{k,T}(\cdot) = \Lambda_{k,N,T}(\cdot)$ almost surely for every $N \in \mathbb{N}$. To do this, for each $N \in \mathbb{N}$, $T > 0$ and $V \in \mathcal{G}_{n-k+1}$, we define the random variable $\xi_N(\omega)$

as follows:

$$\xi_N(\omega) := \sum_{i=0}^N \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|.$$

Then the random variable ξ_N has finite second moment. This implies

$$\limsup_{m \rightarrow \infty} \frac{1}{mT} \xi_N = 0 \quad \text{almost surely.} \quad (3.9)$$

By (2.6) and the fact that $P(s)P_{\text{can}}(s) = P(s)$ for $s \in \mathbb{R}^+$, we get

$$\Psi_{0,t}(\omega) = \Psi_{s,t}(\omega)(\Phi_{0,s}(\omega)P(0)). \quad (3.10)$$

Therefore,

$$\begin{aligned} \Lambda_{k,T}(\omega) &= \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &= \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} (\xi_N(\omega) + \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\|) \\ &\stackrel{(3.9)}{=} \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &\stackrel{(\text{Lemma 3.4})}{=} \inf_{V \in \mathcal{G}(0)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &\stackrel{(3.10)}{=} \inf_{V \in \mathcal{G}(0)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \\ &\quad \times \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\Phi_{0,NT}(\omega)P(0)V)}\| \\ &\stackrel{(3.2)}{=} \inf_{U \in \mathcal{G}(NT)'_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)U}\| \\ &\stackrel{(\text{Lemma 3.4})}{=} \inf_{V \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=N+1}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{NT,iT}(\omega)V}\| \\ &\stackrel{(3.3)}{=} \Lambda_{k,N,T}(\omega). \end{aligned}$$

Consequently, for $k \in \{1, \dots, r\}$ and $T > 0$, we have $\Lambda_{k,T}(\cdot) = \Lambda_{k,N,T}(\cdot)$ almost surely for every $N \in \mathbb{N}$. Combining this result with Lemma 3.6, we conclude that $\Lambda_{k,T}(\cdot)$ is \mathcal{F}_{NT}^∞ -measurable for every $N \in \mathbb{N}$. Consequently, the random variable $\Lambda_{k,T}(\omega)$ is measurable with respect to the tail σ -algebra $\bigcap_{N \in \mathbb{N}} \mathcal{F}_{NT}^\infty$. Since the

Brownian motion has independent increments, the tail σ -algebra $\bigcap_{N \in \mathbb{N}} \mathcal{F}_{NT}^\infty$ is trivial by Kolmogorov's zero-one law (see [2, p. 547]). This implies that the random variable $\Lambda_{k,T}(\omega)$ is degenerate, i.e. nonrandom. On the other hand, applying Lemma 3.5, we obtain

$$\begin{aligned}\Lambda_k(\omega) &= \lim_{T \rightarrow \infty} \inf_{V \in \mathcal{G}_{n-k+1}} \frac{1}{mT} \limsup_{m \rightarrow \infty} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &= \lim_{T \rightarrow \infty} \Lambda_{k,T}(\omega).\end{aligned}$$

Since $\Lambda_{k,T}(\omega)$ are nonrandom for each $T > 0$, we conclude that $\Lambda_k(\omega)$ is also nonrandom. This completes the proof of the theorem. \square

We are now in a position to state and prove the second main result of the paper about a comparison between Lyapunov exponents and central exponents of SDAE (2.2).

Theorem 3.2. *Let $\lambda_k(\omega)$ and $\Lambda_k(\omega)$, $k = 1, \dots, r$, be Lyapunov exponents and central exponents of SDAE (2.2) of index 1, respectively. Then*

- (i) $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_r$.
- (ii) $\Lambda_k \geq \lambda_k$, $k = 1, 2, \dots, r$.

Proof. (i) For $k \in \{1, 2, \dots, r-1\}$, choose and fix an arbitrary $U \in \mathcal{G}_{n-k+1}$. Then, for any $V \in \mathcal{G}_{n-k}$ such that $V \subset U$, we have

$$\|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)V}\| \leq \|\Psi_{s,t}(\omega)|_{\Psi_{0,s}(\omega)U}\|, \quad t \geq s \geq 0.$$

Hence,

$$\begin{aligned}\Lambda_{k+1}(\omega) &= \inf_{V \in \mathcal{G}_{n-k}} \inf_{T \in \mathbb{R}^+} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &\leq \inf_{\substack{V \in \mathcal{G}_{n-k} \\ V \subset U}} \inf_{T \in \mathbb{R}^+} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)V}\| \\ &\leq \inf_{T \in \mathbb{R}^+} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|.\end{aligned}$$

Since U is arbitrarily chosen from \mathcal{G}_{n-k+1} , we obtain

$$\begin{aligned}\Lambda_{k+1}(\omega) &\leq \inf_{U \in \mathcal{G}_{n-k+1}} \inf_{T \in \mathbb{R}^+} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\| \\ &= \Lambda_k(\omega).\end{aligned}$$

(ii) For $k \in \{1, 2, \dots, r\}$, choose and fix an arbitrary $U \in \mathcal{G}_{n-k+1}$. Then, for any $T > 0$, $m \in \mathbb{N}^*$ and $x \in U_*$, we have

$$\begin{aligned} \|\Psi_{0,mT}(\omega)x\| &\stackrel{(2.6)}{=} \|\Psi_{(m-1)T,mT}(\omega)\Psi_{0,(m-1)T}(\omega)x\| \\ &\leq \|\Psi_{(m-1)T,mT}(\omega)|_{\Psi_{0,(m-1)T}(\omega)U}\| \|\Psi_{0,(m-1)T}(\omega)x\| \\ &\leq \|\Psi_{(m-1)T,mT}(\omega)|_{\Psi_{0,(m-1)T}(\omega)U}\| \dots \|\Psi_{0,T}(\omega)|_U\| \|x\|. \end{aligned}$$

Therefore,

$$\sup_{U \in U_*} \limsup_{m \rightarrow \infty} \frac{1}{mT} \log \|\Psi_{0,mT}(\omega)x\| \leq \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|.$$

From this inequality, it follows that

$$\begin{aligned} \inf_{U \in \mathcal{G}_{n-k+1}} \sup_{x \in U_*} \limsup_{m \rightarrow \infty} \frac{1}{mT} \log \|\Psi_{0,mT}(\omega)x\| \\ \leq \inf_{U \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|. \end{aligned}$$

Since the left-hand side is the Lyapunov exponent λ_k , we get for any fixed $T > 0$,

$$\lambda_k \leq \inf_{U \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\|.$$

Consequently,

$$\lambda_k \leq \inf_{T \in \mathbb{R}^+} \inf_{U \in \mathcal{G}_{n-k+1}} \limsup_{m \rightarrow \infty} \frac{1}{mT} \sum_{i=0}^{m-1} \log \|\Psi_{iT,(i+1)T}(\omega)|_{\Psi_{0,iT}(\omega)U}\| = \Lambda_k.$$

The theorem is proved. \square

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